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Inversion for non-smooth models with physical bounds

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Summary

Geological processes produce structures at multiple scales. A discontinuity in the subsurface can occur due to layering, tectonic activities such as faulting, folding and fractures. Traditional approaches to invert geophysical data employ smoothness constraints. Such methods produce smooth models and thefore sharp contrasts in the medium such as lithological boundaries are not easily discernible. The methods that are able to produce non-smooth models, can help interpret the geological discontinuity. In this paper we examine various approaches to obtain non-smooth models from a finite set of noisy data. Broadly they can be categorized into approaches: (1) imposing non-smooth regularization in the inverse problem and (2) solve the inverse problem in a domain that provides multi-scale resolution, such as wavelet domain. In addition to applying non-smooth constraints, we further constrain the inverse problem to obtain models within prescribed physical bounds. The optimization with non-smooth regularization and physical bounds is solved using an interior point method. We demonstrate the applicability and usefulness of these methods with realistic synthetic examples and provide a field example from crosswell radar data.

Introduction

Given a finite amount of noisy data a common practice in many inversion algorithms is to apply smoothness constraints. The goal is largely driven by the desire to obtain simplest model that can explain the data – the Occam's razor principle. Regularization technique such as Tikhonov's approach uses a combination of first or second derivative of the model to impose flatness or smoothness constraint subject to fitting the data (Tikhonov and Arsenin, 1977; Oldenburg, 1990). Typically in the smoothness world an L2 norm of the model derivatives are penalized.

Smooth regularization techniques using L2 norm have certainly produced very stable results when faced with inadequate noisy data to produce simple geologic scenario. In addition, the objective functions designed with L2 norm are computationally easier to solve since their derivative (in the optimization process) produces a linear system of equations. One attractive feature of the smoothness constraint is that it allows us to capture the large length scale behavior of the model. However, earth properties are not always smooth and sharp jumps in material property is a geologic reality. Sharp jumps such as distinct layering or formation of localized bodies in the subsurface can occur due to various geological processes. Examples include different episode of sedimentation to produce layering, hydrothermal alterations to form mineral bodies, structural entrapment of hydrocarbons in a localized region of the subsurface, flow processes that produces sharp fronts; these distinct jumps are realistic outcome of geologic processes. Fundamentally these processes in the subsurface produce multi-scale structures and it is likely that these scales have long range correlations. Thus inversion methods that preserve non-smooth nature of the model are attractive from a practical point of view.

The research for inverting non-smooth models from noisy data have broadly evolved in two directions: (a) impose non-smooth regularization operators in the inversion algorithm and (b) chose appropriate basis functions that effectively honors the sharp discontinuity of the models. In the first category, non-smooth regularization operators such as Lp norm (Oldenburg et. al, 1983; Sacchi and Liu, 2005; Farquharson and Oldenburg, 1998; Routh et. al., 2003), total variation regularization (Bertere-Aguirre et. al, 2002; Farquharson and Oldenburg, 1998) and compactness operators (Last and Kubik, 1983; Portniaguine and Zhadanov, 1999; Ajo-Franklin and Minsley, 2005) have demonstrated much success in many geophysical inversion problems. In this paper in addition to the non-smooth regularization penalty we impose physical bounds on the model to obtain meaningful solutions.

In the second category, the inverse problem is solved in a different domain so that the projection of the model onto the basis functions in this domain preserves the discontinuity and honors the multiscale nature of the model. Wavelet basis is a natural choice due to its localization property and the ability to represent the model with a sparse representation. Several geophysical applications of inversion in the wavelet domain have shown its usefulness (Li et. al., 1996; Kane et. al., 2001; Kane and Herrmann, 2002; Zhu and Li, 2004; Qu and Routh, 2004). In this paper we formulate our inverse problem in the wavelet domain using a mixed norm criteria and solve for a sparse set of wavelet coefficients.

Inversion with Non-Smooth Regularization Operators

In many inverse problems, models are piecewise smooth separated by lower dimensional interfaces. For these problems, it is important to choose a regularization technique that will respect and preserve the discontinuities of the function values, as well as control the geometric regularity of the interfaces. Notable examples of this class of regularization include minimizing the Lp norm, total variation (TV) of the function (Vogel and Oman, 1996) or compact representation of the model by minimum support functions. Consider the inverse problem where we minimize the objective function given by

$$\min \phi = \beta\, R(m) + \left\| W_d \left(d^{obs} - g(m) \right) \right\|^2 \quad , \quad (1)$$

where the model objective function is denoted by $R(m)$ and the second term is the data misfit between observed data d^{obs} and predicted data $g(m)$ and W_d is a data weighting matrix. There can be several choices of the model objective function $R(m)$ to produce non-smooth models. For example, the Lp norm regularization functional is denoted by

$$R(m) = \left(\int |m|^p dv \right)^{\frac{1}{p}} \quad , \quad (2)$$

where $p = 1$ denotes L1 norm. The total variation (TV) regularization is given by

$$R(m) = \int |\nabla m| \; dv \cdot \quad (3)$$

And for minimum support constraints it is given by

$$R(m) = \int \frac{m^2}{m^2 + \alpha} dv \quad , \quad (4)$$

where the integration is carried over the model space. In all three cases the inverse problem (even for a linear forward map) is nonlinear and is solved iteratively. Different variants of eqs (2), (3) and (4) are used in practical implementation to avoid instabilities. The Lp type regularization in eq(2) can be solved using iterative re-weighted least squares (IRLS) approach where the model from the previous iteration is used to weigh the model perturbations (Farquharson and Oldenburg, 1998). Most commonly used norm in this category is an L1 norm which has the ability to produce a sparse model. This is commonly used in reflectivity inversion or deconvolution problem. The TV regularization in eq(3) can also be solved using IRLS procedure where we replace the model with its derivative. Thus if one has an algorithm to solve eq(2), eq(3) can also be implemented. However, the details of how IRLS is implemented need to be tuned. The TV regularization is quite common in the image processing community especially in denoising problems. In a strict sense the bounded variation is given by $|\nabla m|$; however to make the TV functional differentiable a small threshold parameter γ^2 is added. Thus the TV objective function in eq(2) is approximated by $R(m) = \sqrt{|\nabla m|^2 + \gamma^2}$. The minimization of the total objective function for a linear forward problem leads to the solution of the following equation

$$G^T W_d^T W_d G m - G^T W_d^T W_d d^{obs} + \beta \nabla \cdot \left(\frac{\nabla m}{\sqrt{|\nabla m|^2 + \gamma^2}} \right) = 0 \quad (5)$$

Equation (5) is a nonlinear system which can be solved iteratively using a gradient based search technique (Bertere-Aguirre et. al, 2002). Figure 1 clearly shows the advantage of using TV regularization. It is easier to segment and interpret the image in the bottom panel in Figure 1 compared to the smooth model. In the hydrology problem, TV regularization will be valuable in identifying the zones of distinct litho-facies. The minimum support regularization function in eq(4) is well suited to produce compact models commonly encountered in potential field problem (Last and Kubik, 1983; Minsley et. al, 2005) and in time lapse problems where small localized changes in physical propertes are important (Ajo-Franklin and Minsley, 2005). The practical implementation of this functional can also be carried out using IRLS method. A variant of this approach was introduced by Portniaguine and Zhadanov (1999) where they use the gradient of model in eq(4) instead of the model m. This produces minimum support or compactness for the gradient of model with the ability to produce sharp images.

The physical bounds i.e. upper and lower bounds on the model parameters ($m^{MIN} \le m \le m^{MAX}$) are introduced using a primal interior point method (Li and Oldenburg, 2000; Routh et. al, 2005). Thus in addition to minimizing eq(1) we also minimize a log barrier function. Thus the objective function to be minimized is given by

$$\min \phi = \beta\, R(m) + \left\| W_d \left(d^{obs} - g(m) \right) \right\|^2 + \phi_B \quad , \quad (6)$$

where ϕ_B is given by

$$\phi_B = -\lambda \sum_{k=1}^{M} \left(1 - \frac{m_k}{m_k^{MAX}} \right) - \lambda \sum_{k=1}^{M} \left(\frac{m_k - m_k^{MIN}}{m_k^{MAX}} \right) \cdot \quad (7)$$

Inversion with Wavelet Basis

Wavelet-based methods have become a powerful tool to deal with inhomogeneous objects since they can adapt to various ranges of unknown degrees of smoothness. The particular attraction of wavelet-based methods lies in the simple and efficient implementation using Discrete Wavelet Transform (DWT). The most important feature of the wavelet-based methods is the sparse representation of functions expanded in the orthonormal wavelet bases. That is, for a function belonging to a very wide function space, such as Besov space, the vast majority of the wavelet coefficients are zero if the function is expanded in terms of the wavelet bases (Donoho, 1995).

The sparseness has more profound implications than smoothness. One of the goals of this paper is to propose a new approach for the linear inverse problems utilizing the notion of sparseness. By penalizing the L1 norm of the wavelet coefficients of m, we get a sparse solution of m in the wavelet domain. Consequently, this gives us an improved estimate of m compared to the approach based on smoothness for certain function classes. Donoho presented wavelet-vaguelet (WVD) decomposition for solving a class of linear inverse problems in which the kernel is a homogeneous operator such as integration, convolution and radon transform. Under the condition that noise level is small and kernel matrix is invertible, WVD with thresholding offers significant advantage over traditional SVD when the model is non-smooth. Kane et al. (2001) proposed a generalized WVD thresholding by replacing G^{-1} with $\left(G^T G + \beta W_m^{\ T} W_m\right)^{-1} G^T$ in the WVD. They solve the linear inverse problem in wavelet domain. They generalize the traditional L2 norm to Besov norm. Both data misfit and model objective function use the Besov norm. In this approach it is important to determine how one should chose the parameters of the Besov norm.

We pose the inversion in wavelet domain by minimizing the L1 norm of the wavelet coefficients subject to fitting the data using L2 norm. For a linear problem $d = Gm + \varepsilon$ we transform the problem to $d = GW^T Wm + \varepsilon$ where W represents the matrix that performs the wavelet transform. We represent the new system by $d = \tilde{G}\tilde{m} + \varepsilon$ where $\tilde{G} = GW^T$ and $\tilde{m} = Wm$. For discrete wavelet transform $\tilde{m}$ is given by the wavelet coefficients in the expansion

$$m(x) = \sum_k C_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_{j,k}\psi_{j,k}(x) \quad (8)$$

$C_{j_0,k}$ corresponds to weighted mean of the model over the support of scaling function $\phi_{j_0,k}(x)$ at the coarsest scale j_0. The inverse problem in wavelet domain can be written as

$$\min \phi = \beta \left\|D\tilde{m}\right\|_1 + \left\|W_d\left(d^{obs} - \tilde{G}\tilde{m}\right)\right\|^2 \quad (9)$$

where we penalize a L1 norm on the wavelet coefficients. The mixed norm problem is solved using the IRLS procedure. The wavelet weighting matrix D is a diagonal matrix that contains scale dependent weights. Typically the coarse scale information is less penalized compared to fine scale information since the noise is likely to occur in fine scale. Formulating the problem in this manner produces a sparse set of wavelet coefficients that has the ability to capture non-smooth nature of the model. After finding the wavelet coefficients the model is obtained by applying inverse wavelet transform, given by $m = W^T\tilde{m}$.

Tomography Examples

We first present a synthetic study using non-smooth regularization functionals followed by a field example. We consider a crosswell tomography example with the source receiver configuration geometry from a downhole radar field experiment at Boise Hydrogeophysical Research site (BHRS).

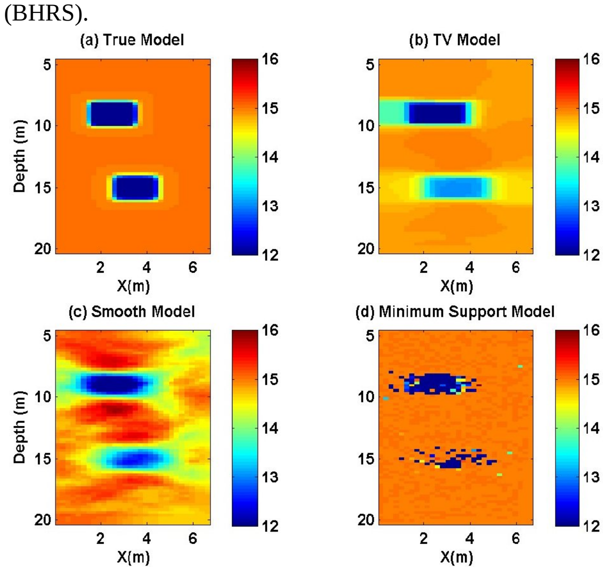

Figure 1: Synthetic tomography example using blocky model to demonstrate the non-smooth regularization inversions. The true model (ns/m) is shown in (a) and the smooth model inversion in (b). (c) represents the model using TV regularization and (d) is the model obtained from minimum support regularization.

The variability in dielectric property beneath the water table at BHRS is small. This somewhat validates using rays to invert the travel time data to obtain slowness distribution. The two block synthetic model shown in Figure 1(a). There are 1700 data values and the medium is discretized into 2160 cells. Synthetic data are contaminated with noise with a standard deviation of 1% of the maximum amplitude of the data.

The model in Figure 1(c) is obtained by imposing smoothness constraint where we penalize the first derivative of the model in x- and z- directions. Although we see the indication of the two blocks, but due to smoothing operator the bodies are not sharply defined. Figure 1(b) shows the result from TV regularization. The recovered model indicates the blocky nature of the model and clutter due to smoothing operator is significantly less. Fig. 1(d) shows the result of minimum support regularization. As expected the minimum support will produce compact bodies that will reproduce the data. Note that we obtain very different models using different regularization schemes. This demonstrates the inherent non-uniqueness of the inverse problem and implies that depending on what kind of model is expected from geological consideration, the choice of appropriate regularization is crucial. We note that TV provides the best representation of a blocky model.

Figure 2 shows the results from field data from the BHRS. Here 3500 data are inverted using three different regularization methods. Figure 2(b) clearly indicate the sharp boundary due to layering in the sedimentary environment at BHRS.

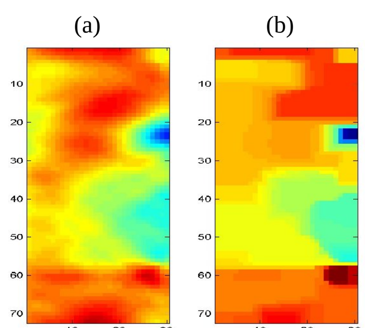

Figure 2: Field example of crosswell radar from Boise Hydrogeophysical Research Site. The smooth model inversion is shown in (a). (b) represents the model using TV regularization. Color scale for slowness is is 9 (blue) -13 (brown) ns/m.

1D Synthetic Example

Using a 1D model with sharp discontinuity we compare results from the wavelet domain inversion with TV regularization and the smooth L2 norm constraints. The top panel in Figure 3 is obtained with signal to noise ratio (SNR) of 8 and the bottom panel is obtained when SNR=2. For SNR=8 we note that the models recovered using all three methods are comparable with wavelet domain and TV model performing slightly better than smooth L2 norm model. However for SNR=2 we clearly see that the wavelet domain inversion shows better representation of the true model compared to TV and L2 norm models. The wavelet chosen to perform the inversion in wavelet domain is a Haar wavelet.

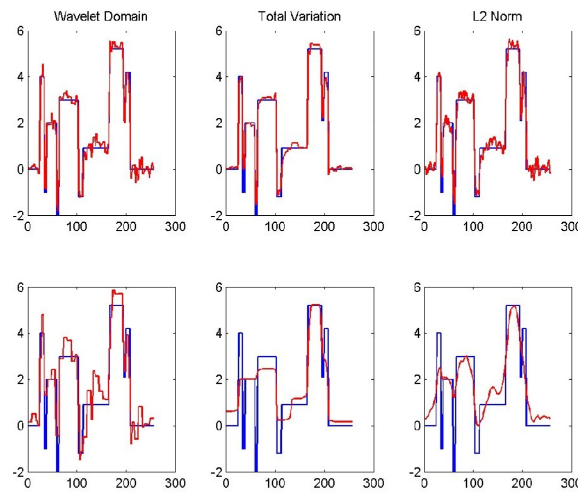

Figure 3: 1D Synthetic example with sharp discontinuity to compare the wavelet domain inversion with other methods. The top panel are the models using SNR=8 and the bottom panel is with SNR=2. In all panels blue is the true model and red is the inverted model.

Conclusions

In this paper we discuss various techniques to image sharp discontinuities in a model using non-smooth regularization operators and inversion in wavelet domain by scale based regularization of wavelet coefficients. We introduce physical bounds on solutions using an interior point method to obtain realistic solutions. Synthetic and field examples presented in this paper demonstrate significant improvement in the inverted models compared to those obtained with traditional smoothness based inversion.

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EDITED REFERENCES

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